II. *The Doctrine of Combinations and Alternations, Improv'd and Compleated, by Major* Edward Thornycroft.

IN order to underſtand what follows, it muſt be obſerved,

1ſt, That as in the notation of Powers, a a a a b b b c c is deſign'd by a^4 b^3 c^2, and univerſally p times the poſition of a, q times the poſition of b, r times the poſition of c, by a^p b^p c^r, ſo in things expos'd likewiſe: (unleſs where 'tis propos'd they ſhould be all different) which Indices, as they have here no relation to Powers, but expreſs only the Occurrences of thoſe things to which they reſpectively belong; I therefore call Indices of Occurrences.

2dly, That as often as I ſhall hereafter mention the Combinations or Alternations of the p^s q^s r^s or s^s, (which conſider'd by themſelves are capable of no variation) I mean of thoſe things whoſe Indices they are.

3dly, That m is generally put for the whole number of things expos'd, whether all different or not, *i. e.* equal to the ſum of their Indices; and n, for ſuch a number of them, as each Combination and Alternation muſt conſiſt of; (unleſs preſuppoſed equal) which explains what is hereafter meant by the Combinations and Alternations of m things taken n and n; or of m things taken m and m; and the like Expreſſion, by whatever Symbols the number of things out of which the Combinations and Alternations are to be made, or of which they are to conſiſt, may be deſign'd.

Lemma 1st.

If in a right Line, at any distances, be plac'd any number of things, a b c d, *&c.* the number of the Intervals a b, b c, c d, *&c.* terminated each by two adjacent things, is one less than the number of things.

For, whereas every Interval is terminated by two adjacent things, if to any number of things, be added one thing more, one Interval only is thereby added. Q. E. D.

Lemma 2d.

The number of the Alternations of m things a b c d, *&c.* different each from other, taken m and m, is m times the number of the Alternations of m—1 things a b c, taken m—1 and m—1.

For, (by *Lem.* 1st) the last Letter d, besides the position it hath, may have m—2 positions, *viz.* in the Intervals which are between m—1 things a b c; but it may also have one more, for it may be put first of all, it may therefore have m positions; and those in all the different Orders, whereof m—1 things are capable; which being all the possible positions of d, in all the varieties of a b c, is all the variety whereof the whole number of things exposed a b c d, *&c.* is capable. Q. E. D.

Lemma 3d.

The number of the Alternations of m things a b c d, *&c.* different each from other, taken m and m, is equal to $m \times \overline{m-1} \times m\,\overline{m-2} \times \overline{m-3} \times \overline{m-4}$, *&c.* continu'd to m places.

For let m O, express the number of the Alternations of m things different each from other; $\overline{m-1}$ O, of m—1 things and the like.

'Tis evident that if m = 1, It will be m O = m; for there can be but one order of one thing.

And if m be greater than unity, then will it be (by *Lem.* 2) $m\,O = m \times \overline{m-1}\,O = m \times \overline{m-1} \times \overline{m-2}\,O = m \times \overline{m-1} \times$

$\overline{m-2}$

$m-2 \times \overline{m-3}\,O =$, *&c.* till we have an Equation consisting of m places; *i. e.* $= m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times$ *&c.* continu'd to m places. Q. E D.

Lemma 4th.

If $m\,\omega$ Express the number of the Alternations of m things $a^p\, b^p\, c^p\, d^q\, e^q\, f^r$ *&c.* taken m and m, and α the number of p's, β the number of q's, γ the number of r's, it will be

$$m\,\omega = \frac{m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \overline{m-4} \times \overline{m-5} \times \text{\&c. contiu'd to m places.}}{\overline{p \times p-1 \times p-2 \times \&c.}^{\alpha} \times \overline{q \times q-1 \times \&c.}^{\beta} \times \overline{r \times r-1 \times \&c.}^{\gamma}}$$ each Series continued to

p, q, r, *&c.* places respectively.

For the number of the Alternations of any number of things, however divided into parts, is produc'd by a continual Multiplication of the Alternations of those things amongst themselves respectively, which compose each part, into the number of their Alternations one amongst the other; *i. e.* in the present case (the several occurrences being supposed to compose the several parts, and consequently the number of the Alternations of the things composing each part equal to unity) $m\,\omega =$ to the number of the Alternations of the things composing the parts one amongst the other; but the number of their Alternations one amongst the other, is the same in this case, as if the things expos'd, being all different, were divided into the same parts; for the things which compose each part in both cases, are different from the rest of the things expos'd; *i. e.* by *Lem.* 3d.

$$m\,\omega = \frac{m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \overline{m-4} \times \overline{m-5} \times \text{\&c. continued to m places}}{\overline{p \times p-1 \times p-2 \times \&c.}|^{\alpha} \times \overline{q \times q-1 \times \&c.}|^{\beta} \times \overline{r \times r-1 \times \&c.}|^{\gamma}}$$ each Series continued

to p, q, r places respectively. Q. E. D.

Lemma 5th.

The number of the Combinations of m things a b c d, *&c.* different each from other, taken n and n, is equal to $\frac{m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \&c.}{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \&c.}$ each Series continued to n places

For if the things expos'd be divided in two parts, *viz.* in the *ratio* of n and m—n, 'tis evident that their different binations taken n and n, are produc'd by the Alternations of the things composing the parts one amongst the other: And therefore the number of those = to the number of these = to the number of the Alternations of m things taken m and m, the Indices of whose occurrences are n and m—n $= \frac{m \times m-1 \times n-2 \times m-3 \times \&c. \text{ continued to m places}}{n \times n-1 \times \&c. \times m-n \times m-n-1 \times \&c. \text{ each Series continued to n and m-n}}$ places respectively (by *Lem.* 4th) *i. e.* because n + m—n $= m = \frac{m \times m-1 \times m-2 \times m-3 \&c.}{n \times n-1 \times n-2 \times n-3 \&c.}$ each Series continued to n places Q. E. D.

But the number of the Alternations in every Combination is $= n \times n-1 \times n-2 \times n-3 \times$, *&c.* continued to n places, by *Lem.* 3d therefore.

Lemma 6th.

The number of the Alternations of m things a b c d, *&c.* different each from other, taken n and n, is $= m \times m-1 \times m-2 \times m-3 \times$, *&c.* continued to n places. Q. E. D.

Scholium.

Since in the things expos'd the same things may occur more than once, and also n be less than m, the Indices of the occurrences which are in some of the Combinations of m things taken n and n, may differ from those which are in others; but those Combinations, the Indices of whose occurrences are the same, are said to be in the same form: Therefore whereas n is equal to the sum of the Indices which are in each Combination taken n and n, if n be express'd by all the different Combinations of such Indices only (being integer numbers) whereof no one may exceed the highest Index of the things expos'd, and being more than one in a Combination, are each of them, which are in the same Combination, comprehended in a distinct Index thereof; these Expressions of n will necessarily be the several forms of the Combinations taken n and n, whereof m things are capable: Whence is deriv'd a General

The.

Theorem for finding the Combinations and Alternations of m things taked n and n universally: *i. e.* Whether m consist of things all different or not, and whether n be equal to, or less than m.

Theorem.

If n be express'd, according to all the different Forms of Combination which the things expos'd are capable of,

and { p = the highest Index; q = the next highest; r = the next highest; ſ = the next highest; &c. } { α = the number of p's; β = the number of q's; γ = the number of r's; δ = the number of ſ's } In every form of Combination.

{ A = the number of all Indices not less than p; B = the number of all Indices not less than q; C = the number of all Indices not less than r; D = the number of all Indices not less than ſ; &c. } Which are in the things Expos'd

and b $= \alpha + \beta$, c $= b + \gamma$, d $= c + \delta$, &c.

I say the number of the Combinations of m things taken n and n, in any one Form of Combination, shall be

$$\frac{A \times \overline{A-1} \times \overline{A-2}\ \&c.}{\alpha \times \overline{\alpha-1} \times \overline{\alpha-2}\ \&c.} \times \frac{\overline{B-\alpha} \times \overline{B-\alpha-1}}{\beta \times \overline{\beta-1}} \times \&c. \times \frac{\overline{C-b} \times \overline{C-b-1}}{\gamma \times \overline{\gamma-1}} \times \&c. \times \frac{\overline{D-c} \times \overline{D-c-1}}{\delta \times \overline{\delta-1}}\ \&c.$$

continued to so many Terms as there are different Indices in the form of Combination and each Term to α, β, γ, δ, &c. places respectively, and this number multiply'd into

$$\frac{\times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-4} \times \overline{n-5} \times \overline{n-6}\ \&c.\ \text{continued to n places,}}{\overline{p \times \overline{p-1} + \overline{p-2} \times \&c.}\,|^{\alpha} \times \overline{q \times \overline{q-1} \times \&c.}\,|^{\beta} \times \overline{r \times \overline{r-1} \times \&c.}\,|^{\gamma} \times \&c.}$$

each Series continued to p, q, r, &c. places respectively, shall be the number of their Alternations.

But the sum of all the Combinations and Alternations which are in every Form of n, shall be the whole number of Combinations and Alternations of m things taken n and n.

De-

Demonstration.

First, Then 'tis evident, that those Combinations, which are in different forms, differ from each other.

Again, 'Tis evident that the Combinations of m things, as $a^p\ b^p\ c^p\ d^p\ e^q\ f^q\ g^q\ h^r\ i^r$, &c. (the Indices simply consider'd) taken n and n, in a form wherein are p^s q^s and r^s, shall be equal to the number of the Combinations of the p^s, which are in the things expos'd, taken α and α, multiply'd into the number of the Combinations of the q^s taken β and β multiply'd into the number of the Combinations of the r^s taken γ and γ.

But because p and all lesser Indices are comprehended in every Index which is greater than themselves; therefore is A = to the number of p^s which are in the things expos'd, and for the same reason, would B = the number of the q^s, and C the number of r^s: But the number of the p^s, which are in every form of Combination, is = α; therefore is B — α = to the number of q^s; also because the number of p^s and q^s together, which are in every form of Combination, wherein there are q^s, is = $\alpha + \beta$ = b; therefore is C — b = to the number of r^s, and so on, how many soever were the different Indices in any form of Combination.

But (by *Lem.* 5th) the number of the Combinations of the p^s, which are in the things expos'd, whose number is A, taken α and α, is = $\frac{A \times A-1 \times A-2}{\alpha \times \alpha-1 \times \alpha-2}$ &c. continu'd to α places, and the number of the Combinations of the q^s, whose number is B—α, taken β and β, is = $\frac{B-\alpha \times B-\alpha-1 \times B-\alpha-2}{\beta \times \beta-1 \times \beta-2}$ &c. continued to β places, and the number of the Combinations of the r^s, whose number is C—b, taken γ and γ, is = $\frac{C-b \times C-b-1}{\gamma \times \gamma-1}$ &c. continued to γ places Q. E. D.

But

But every Combination, in one and the same form, affords the same number of Alternations: Therefore the number of Alternations, in any one form, is so many times the number of Combinations, as is the number of Alternations in any one of these Combinations.

But (by *Lem.* 4th) the number of Alternations in any of those Combinations shall be

$$\frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5} \times \overline{n-6} \times \&c. \text{ continued to n places}}{\overline{p \times \overline{p-1} \times \overline{p-2} \times \&c.}|^{\alpha} \times \overline{q \times \overline{q-2} \times \&c.}|^{\beta} \times \overline{r \times \overline{r-1} \times \&c.}|^{\gamma} \times} \text{ each Series continued to p q r \&c.}$$

places respectively. Q. E. D.

Now to make an application of this general Rule to those particular cases which have already been consider'd by others, and which are contain'd in our 3d, 4th, 5th and 6th *Lemma's*, and by us more generally demonstrated; I say

If n = m, there can be but one form of Combination, and but one Combination in that form; and therefore the number of Alternations $= \dfrac{m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \overline{m-4} \times \&c. \text{ contin'd to m places}}{\overline{p \times \overline{p-1} \times \&c.}|^{\alpha} \times \overline{q \times \overline{q-1} \times \&c.}|^{\beta} \times \overline{r \times \&c.}|^{\gamma} \times}$ *&c.* each Series to p q r, *&c.* places respectively, *i. e.* (if p = 1) $= m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \overline{m-4} \times$ *&c.* continu'd to m places, which are the cases of the 4th and 3d *Lemma's*.

But if the things expos'd are all different, and n be less than m, which is the case of the 5th and 6th *Lemma's*, then also can there be but one form of Combination, and it will be A = m & α = n, and the whole number of Combinations $= \dfrac{A \times \overline{A-1} \times \overline{A-2} \times \&c.}{\alpha \times \overline{\alpha-1} \times \overline{\alpha-2} \times \&c.}$ *i. e.* $= \dfrac{m \times \overline{m-1} \times \overline{m-2} \times \&c}{n \times \overline{n-1} \times \overline{n-2} \times \&c}$ each Series continued to n places, and therefore the number of Alternations $= m \times \overline{m-1} \times \overline{m-2} \times$ *&c.* continu'd to n places.

But fully to Illustrate this Theorem, which, as deliver'd in general, may seem somewhat too Abstracted, to be commonly understood, I shall subjoyn one short Example.

Ex-

Example.

Let the things expos'd be a a a b b b c c, or according to our way of notation $a^3\, b^3\, c^2$; 'Tis requir'd to find the number of their Combinations and Alternations taken 4 and 4.

Then (becaufe in the things expos'd, there is no one thing occurs more than thrice, nor more than three things different from each other) will all the forms of Combination, which the things expos'd are capable of, be thefe,

$$viz. \left\{\begin{matrix} 3\ .\ 1 \\ 2\ .\ 2 \\ 2\ .\ 1\ .\ 1 \end{matrix}\right\} \text{Then}$$

In the 1ft form will p = 3, q = 1, $\alpha = 1$, $\beta = 1$, A = 2, B = 3

In the 2d form will p = 2, ——, $\alpha = 2$, ——, A = 3, ——

In the 3d form will p = 2, q = 1, $\alpha = 1$, $\beta = 2$, A = 3, B = 3

The number of Combinations in the 1ft form $= \frac{A}{\alpha} \times \frac{\overline{B-\alpha}}{\beta} = \frac{2}{1} \times \frac{2}{1} = 4$

The number of Combinations in the fecond form $= \frac{A \times \overline{A-1}}{\alpha \times \overline{\alpha-1}} = \frac{3 \times 2}{2 \times 1} = 3$

The number of Combinations in the 3d form $= \frac{A}{\alpha} \times \frac{\overline{B-\alpha} \times \overline{B-\alpha-1}}{\beta \times \overline{\beta-1}} = \frac{2 \times 1}{2 \times 1} = 3$

And the whole number of Combinations = 10

Alfo the number of Alternations.

In the 1ft form $= 4 \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{\overline{p \times \overline{p-1} \times \overline{p-2}}|^{\alpha} \times q^{\beta}} = 4 \times \frac{4 \times 3 \times 2 \times 1}{\overline{3 \times 2 \times 1}|^{1} \times 1} = 4 \times 4 = 16$

In the 2d form $= 3 \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{\overline{p \times \overline{p-1}}|^{\alpha}} = 4 \times \frac{4 \times 3 \times 2 \times 1}{\overline{2 \times 1}|^{2}} = 3 \times 6 = 18$

In the 3d form $= 3 \times \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{\overline{p \times \overline{p-1}}|^{\alpha} \times q^{\beta}} = 3 \times \frac{4 \times 3 \times 2 \times 1}{\overline{2 \times 1}|^{1} \times 1^{2}} = 3 \times 12 = 36$

And the whole number of Alternations = 70

Many are the Properties of this *Theorem*, in common with others, as, To find the *Unciæ* of a Multinomial rais'd raiſ'd to any integer power. To raiſe an Infinite Series to an integer power, though of an interrupted Order, without introducing any thing immaterial, or which muſt afterwards be expung'd; and many others. But then ſo many Terms of the Series muſt be taken in at firſt as ſhall ſerve to the purpoſes of the intended approximation, otherwiſe as often as it ſhall fall ſhort of that, the Operation muſt be began *de novo.*

Many likewiſe are the Properties peculiar to this *Theorem,* and great Variety of Problems might be fram'd; and I ſcruple not to ſay, many may occur in Practice, which are ſolvable by this, and no other Method whatever.

Hence may be found the number of Words whereof the 24 Letters are capable, from one Letter in each Word, to any number of Letters given.

Hence may be found the number of all Numbers, to any given number of Places, which may be produc'd from any number of Figures given.

Hence alſo the Compaſs of a Muſical Inſtrument being given, the Time and number of the Bars, whereof each Tune ſhall conſiſt, the number of Tunes may be found which that Inſtrument is capable of.

To give an inſtance of the prodigious variety that there is in Muſick, I have Calculated the number of Tunes in Common Time, conſiſting of eight Bars each, which may be play'd on an Inſtrument of one Note Compaſs only, and it is this, *viz.* 27584. 270157. 013570. 368586. 999728. 299176. whereas the Changes on 24 Bells is but 620448. 401733. 239439. 360000, which is but $\frac{1}{444588.\ 604583}$ of the number of Tunes, and yet Dr *Wallis* in his *Algebra* demonſtrates, could not be diſpatch'd in 31557. 600000. 000000 years.

If then the Inſtrument were of as many Notes Compaſs as any Inſtrument now in uſe, how prodig[illegible] number of Tunes be encereas'd; the Calculation of which (tho much more intricate and operose) would be equally attainable by our Theorem.

III. *Of Ossifications or Petrifactions in the Coats of Arteries, particularly in the Valves of the Great Artery, by* William Cowper, *Surgeon, and F. R. S.*

HOw far Anatomical Enquiries inform in the true causes of Diseases, which have been ascribed to the want of Spirits in some, and Radical Moisture in Aged People, *&c.* may be in some measure seen by two Observations, among others, publisht in the Transactions No 280: The first there mentioned, pag. 1195, is of a young Gentlewoman, in whom the *Parietes*, or Membranes, that compose the Trunks of the Arteries of the Arm near the *Axilla*, being very much thickened, so that the *Diameter* of its Bore was lessened to more than a third part of its natural size; insomuch that a part of the Trunk of the Artery cut Transversel very much resembled a bit of the stem of a Tobacco-pipe, its sides were so thick, and its Bore consequently so much lessened: The other was of the Trunks of the Arteries of the Leg, pag. *ib.* that were Obstructed by Petrifactions or Ossifications, in a person about the 67th year of his Age. Since which I have met with several of the like Instances in people of years, particularly in the Leg of an old Gentleman, whose Toes and Foot were Sphacelated, the Arteries of whose Leg I have still by me, and have sent them herewith Injected, as much as they could be, with Red Wax; in which the Ossifications diminishing their Channels in some places, and totally obstructing them in others,